ON KHINTCHINE INEQUALITIES WITH A WEIGHT

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ABSTRACT. In this note we prove a weighted version of the Khintchine inequalities.

Let $(\Omega, \mathcal{A}, \mathbb{P})$ be a probability space and let $(r_n)_{n\geq 1}$ be a Rademacher sequence. For a random variable $\xi : \Omega \to \mathbb{R}$ and $p > 0$ we write $\|\xi\|_p = (\mathbb{E}|\xi|^p)^{1/p}$. Our main result is the following weighted version of Khintchine's inequality. We also allow the weight to be zero on a set of positive measure.

Theorem 1. *Let $0 < p < \infty$ and let $w \in L^q(\Omega)$ for some $q > p$, and assume $s := \mathbb{P}(w \neq 0) > 2/3$. Let $\xi = \sum_{n\geq 1} r_n x_n$ with $\sum_{n\geq 1} x_n^2 < \infty$. Then there exist constants $C_1 := C_1(p, w), C_2 := C_2(p, w) > 0$ such that*

$$C_1^{-1}\Big(\sum_{n\geq 1} x_n^2\Big)^{\frac{1}{2}} \leq \|w\xi\|_p \leq C_2\Big(\sum_{n\geq 1} x_n^2\Big)^{\frac{1}{2}}. \tag{1}$$

Consequently, the p-th moments for $0 < p < q$ are all comparable.

If $w \equiv 1$ the result reduces the Khintchine inequalities [4]. Although the weighted version of the result is easy to prove, to our knowledge it was not known, and potentially useful for others. We need a well-known L^0-version of Khintchine's inequality. We provide the details to obtain explicit constants.

Proposition 2. *For all $a \in (0, 1)$ and for all $(x_n)_{n\geq 1}$ in ℓ^2, one has*

$$\mathbb{P}\Big(\Big|\sum_{n\geq 1} r_n x_n\Big| > a\Big) \leq (1 - a^2)^2/3 \ \Rightarrow \ \sum_{n\geq 1} |x_n|^2 \leq 1$$

We need the Paley-Zygmund inequality (see [2, Corollary 3.3.2]) which says that for a positive nonzero random variable $\xi : \Omega \to \mathbb{R}$ and $q \in (2, \infty)$ one has

$$\mathbb{P}(\xi > \lambda\|\xi\|_2) \geq \Big[(1 - \lambda^2)\frac{\|\xi\|_2^2}{\|\xi\|_q^2}\Big]^{q/(q-2)} \quad \lambda \in [0, 1].$$

Proof. Assume $\sum_{n\geq 1} x_n^2 > 1$. Let $\xi = \Big|\sum_{n\geq 1} r_n x_n\Big|$ and $m := \|\xi\|_2 > 1$. Recall the following case of Khintchine's inequality: $\mathbb{E}\xi^4 \leq 3(\mathbb{E}\xi^2)^2$ (see [2, Section 1.3]). Therefore, the Paley-Zygmund inequality applied shows that

$$\mathbb{P}(\xi > a) \geq \mathbb{P}(\xi > a\|\xi\|_2) \geq (1 - a^2)^2 \frac{(\mathbb{E}\xi^2)^2}{\mathbb{E}\xi^4} \geq (1 - a^2)^2/3.$$

□

We will also need the following lemma.

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Lemma 3. *Let* $\eta = \sum_{n\geq 1} r_n x_n$*, with* $\sum_{n\geq 1} x_n^2 \in (0,\infty)$*. Then* $\mathbb{P}(\eta = 0) \leq 1 - 2e^{-2+\gamma} \approx 0.517$*, where* γ *is Euler constant.*

Note that for $\eta = r_1 + r_2$ one has $\mathbb{P}(\eta = 0) = 1/2$, which shows that the lemma is close to optimal.

Proof. By scaling we can assume $\|\eta\|_2 = 1$. By the Paley-Zygmund inequality applied with $\xi = |\eta|$ together with the best constant in the Khintchine inequality (see [3]) one sees that for all $\lambda \in (0,1)$ and $q > 2$,

$$\mathbb{P}(|\eta| > \lambda) = \mathbb{P}(\xi > \lambda) \geq \Big[(1-\lambda^2)B_q^{-2}\Big]^{q/(q-2)},$$

where $B_q = \sqrt{2}\Big(\frac{\Gamma((p+1)/2)}{\sqrt{\pi}}\Big)^{1/q}$. An elementary calculation for Γ-functions shows that $B_q^{-2q/(q-2)} \to 2e^{-2+\gamma}$ as $q \downarrow 2$. Now the result follows by first taking $q > 2$ arbitrary close to 2 and then λ small enough. $\square$

Proof of Theorem 1. The second estimate follows from Hölder's inequality with $\frac{1}{p} = \frac{1}{q} + \frac{1}{r}$ and the unweighted Khintchine inequality with constant $k_{r,2}$:

$$\|w\xi\|_p \leq \|w\|_q \|\xi\|_r \leq \|w\|_q k_{r,2} \Big(\sum_{n\geq 1} x_n^2\Big)^{\frac{1}{2}}.$$

Next we prove the first estimate. Since $\|w\xi\|_p$ increases in p, it suffices to consider $p \in (0,2]$. If all the x_n are zero, there is nothing to prove. If not, then by Lemma 3 and the assumption we have $\mathbb{P}(w\xi \neq 0) = \mathbb{P}(w \neq 0, \xi \neq 0) > 0$, and therefore $\|w\xi\|_p > 0$. To complete the proof we can assume that $\|w\xi\|_p = 1$ as follows by a scaling argument. Moreover, by replacing w by $|w|$ if necessary, we can assume that w is nonnegative.

Choose $a \in (0,1)$ so small that $b = (1-a^2)^2/3 > 1 - s$, where $s = \mathbb{P}(w \neq 0)$. (For example take a such that $b = (1-a^2)^2/3 = [(1-s) + 1/3]/2)$. Let

$$\delta_0 = \sup\{\delta > 0 : \mathbb{P}(w > \delta) \geq (s + 1 - b)/2\}.$$

Since $\mathbb{P}(w > 0) = s > (s+1-b)/2$ we have $\delta_0 > 0$. Let $A = \{w \geq \delta_0\}$. Then it follows that for all $t > 0$

$$\begin{aligned}\mathbb{P}(\{|\xi| > t\} \cap A) &= \mathbb{P}(\mathbf{1}_A|\xi| > t) \leq t^{-p}\mathbb{E}(\mathbf{1}_A|\xi|^p) \\ &\leq t^{-p}\delta_0^{-p}\mathbb{E}(w^p\mathbf{1}_A|\xi|^p) \leq t^{-p}\delta_0^{-p}\mathbb{E}(|w\xi|^p) = t^{-p}\delta_0^{-p}.\end{aligned}$$

Therefore,

$$\mathbb{P}(\{|\xi| > t\}) \leq \mathbb{P}(\{|\xi| > t \cap A\} + \mathbb{P}(\Omega \setminus A) \leq t^{-p}\delta_0^{-p} + 1 - (s+1-b)/2.$$

Now with $t = \delta_0^{-1}\Big(b - 1 + (s+1-b)/2\Big)^{-\frac{1}{p}}$ it follows that $\mathbb{P}(\{|\xi| > t\} \leq b$. Let $y_n = \frac{a x_n}{t}$ and $\eta = \sum_{n\geq 1} r_n y_n$. Then $\mathbb{P}(|\eta| > a) = \mathbb{P}(\{|\xi| > t\}) \leq b$. Therefore, Proposition 2 gives that $\sum_{n\geq 1} y_n^2 \leq 1$. In other words $\sum_{n\geq 1} x_n^2 \leq \frac{t^2}{a^2}$ and the result follows with $C_1 = a/t$. $\square$

Remark 4.

(1) A more sophisticated application of the Paley-Zygmund inequality in Proposition 2 shows that in the theorem it suffices to assume that $\mathbb{P}(w \neq 0) > 1 - 2e^{-2+\gamma} \approx 0.517$. This is close to optimal as can be seen by taking $w = \mathbf{1}_{r_1+r_2\neq 0}$ and $\xi = r_1 + r_2$ for which the weighted inequality (1) does not hold.
(2) The integrability condition on w used for the second estimate of (1) can be improved. However, the general function space for w is difficult to describe and not even rearrangement invariant (cf. [1]).
(3) With a similar technique one can obtain Theorem 1 for Gaussian random variables, q-stable random variables, etc.
(4) The case where the x_n take values in a normed space X, can also be considered. Then $\left(\sum_{n\geq 1} x_n^2\right)^{\frac{1}{2}}$ has to be replaced by the L^2-norm $\|\xi\|_2$, where $\xi = \sum_{n\geq 1} r_n x_n$. Note that Lemma 3 extends to this setting, as follows by applying Lemma 3 with $\eta = \langle \xi, x^* \rangle$ for a functional $x^* \in X^*$ for which $\langle \xi, x^* \rangle$ is nonzero. Also the constants in Proposition 2 can be taken as before. This follows from the fact that also in the vector-valued setting $\|\xi\|_4 \leq 3^{1/4}\|\xi\|_2$ (see [5]).

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